Optimization of Polytopic System Eigenvalues by Swarm of Particles

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Plan

- Introduction
  - Eigenvalue optimization
  - Linear polytopic systems
- Problem formulation and features
- Proposed (U)PSO modifications
- Numerical experiments results
- Conclusions
- Discussion

Eigenvalue optimization

- fascinating and continuous challenge
- most popular problem: minimization of maximum real part of system eigenvalues
- applications: physic, chemistry, structural design, mechanics, etc.

Eigenvalue optimization

- eigenvalues of dynamic systems:
  - measure of robustness
  - information regarding stability/instability

Eigenvalue optimization

- symmetric matrices:
  - may be solved e.g. by convex optimization

Blanco, A.M., Bandoni, J.A.: Eigenvalue and Singular Value Optimization. In: ENIEF 2003 - XIII Congreso sobre Métodos Numéricos y sus Aplicaciones, pp. 1256–1272 (2003)

- non-symmetric matrices:
  - non-convex
  - non-differentiable
  - multiple local optima

Linear polytopic system

- system matrix:
  - in a convex hull of vertices: $A_1, ..., A_N$
  - combination coefficients: $\alpha_1, ..., \alpha_N$

$$\frac{dx}{dt} = A(\alpha)x \qquad A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i$$

$$\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_N]^T, \quad \alpha_i \geq 0,\ i = 1,2,\cdots,N \quad \sum_{i=1}^{N} \alpha_i = 1$$

Problem formulation

$$\text{Minimize } f(\alpha) = \max_{1 \le i \le n} \mathrm{Re}\lambda_i\big(A(\alpha)\big)$$

$$\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_N]^T, \;\; \alpha_i \ge 0, \;\; i = 1,2,\cdots, N-1$$

$$\textstyle\sum_{i=1}^{N-1} \alpha_i \le 1, \;\; \alpha_N = 1 - \sum_{i=1}^{N-1} \alpha_i$$

Problem features

- *N* vertices
- *N-1* optimization variables
- stable vertices $\not\Rightarrow$ stable polytopic system

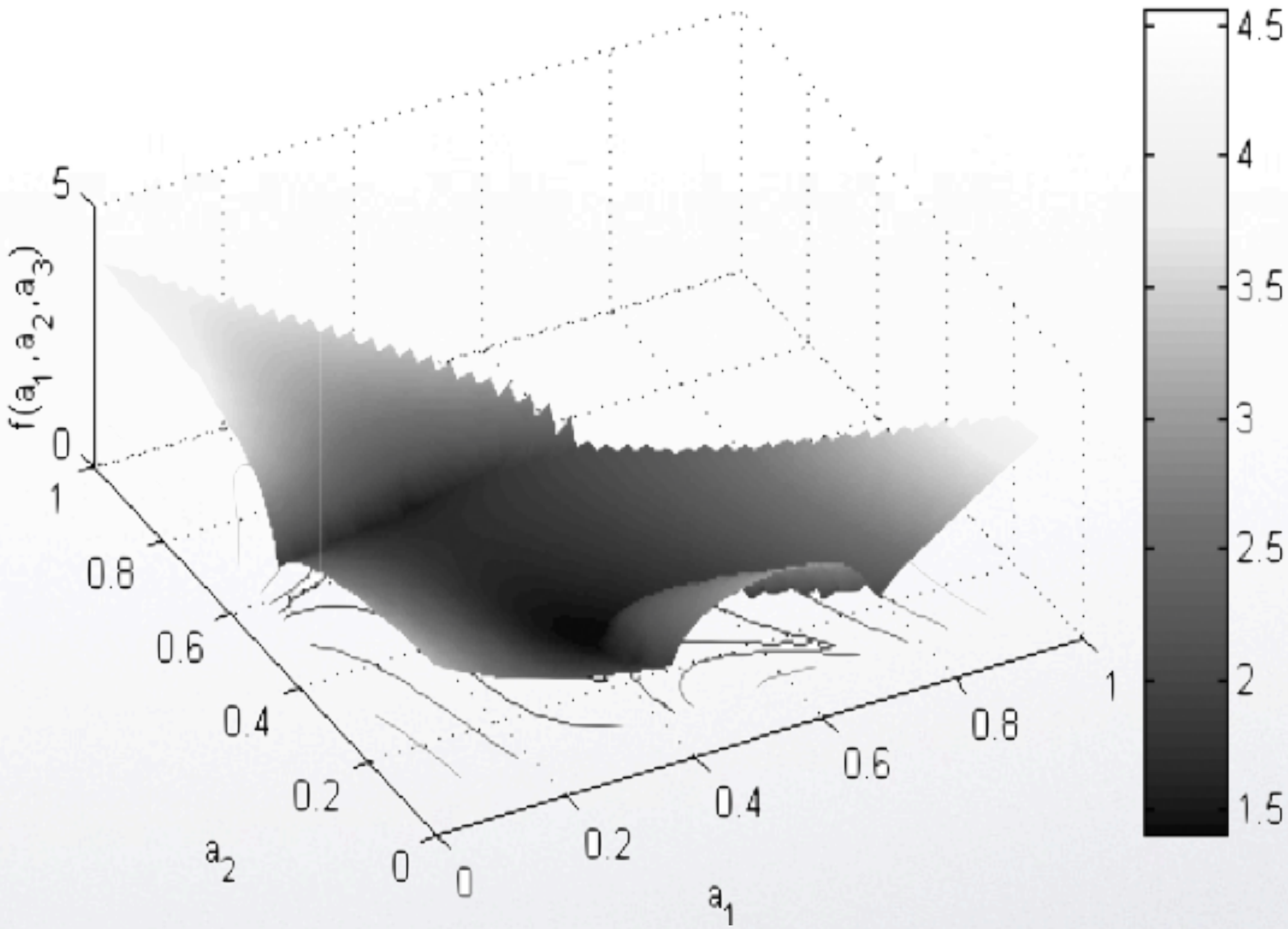
$$\begin{bmatrix} -0.1 & 2 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} -2 & 1.9 \\ 1.9 & -2 \end{bmatrix} \qquad \begin{bmatrix} -1.05 & 1.95 \\ 0.95 & -1.1 \end{bmatrix}$$

0.2863
-2.4363

- non-convex, non-differentiable, several local minima

Problem features

- hard constraints
- global minimum located on boundary or vertex



Particle Swarm Optimization

- stochastic optimization algorithm based on social simulation models
- multiple modifications, selected: UPSO
  - combines exploration and exploitation abilities
  - superiority in terms of success rate and number of function evaluations

Parsopoulos, K.E., Vrahatis, M.N.: UPSO: A unified particle swarm optimization scheme. In: Proc. Int. Conf. Comput. Meth. Sci. Eng. (ICCMSE 2004). Lecture Series on Computer and Computational Sciences, vol. 1, pp. 868–873. VSP International Science Publishers, Zeist (2004)

Proposed modifications (I)

Constraints - initial position in N-1 dimensional simplex

- algorithm:
  - random position in N-1 dimensional unit hypercube $\hat{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_{N-1}]^T$
  - if $\sigma = \sum_{i=1}^{N-1} \alpha_i > 1$

    divide $\hat{\alpha}$ by random coefficient bigger than σ :

$$\hat{\alpha} := \frac{\hat{\alpha}}{\sigma(1+r)}$$

Proposed modifications (2)

Constraints:

- hard - any point outside simplex is infeasible
- global minimum often on simplex face or vertex

- modification: ability to slide along face or edge

Proposed modifications

Algorithm:

- previous position: $\hat{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_{N-1}]^T$
- new position: $\hat{\beta} = [\beta_1, \beta_2, \cdots, \beta_{N-1}]^T$
- calculate minimum distance t_{min} from $\hat{\beta}$ to boundaries:

$$\sum_{i=1}^{N-1} \alpha_i = 1 \qquad \alpha_i = 0$$

- replace new position with intersection point:

$$\widehat{\alpha'} = \widehat{\beta} + t_{\min} \cdot v$$

- neutralize velocity component orthogonal to boundary hyper-plane:
  - for $\alpha_i = 0$ - appropriate component zeroed
  - for $\sum_{i=1}^{N-1} \alpha_i = 1$ - velocity set to parallel component

Numerical experiments

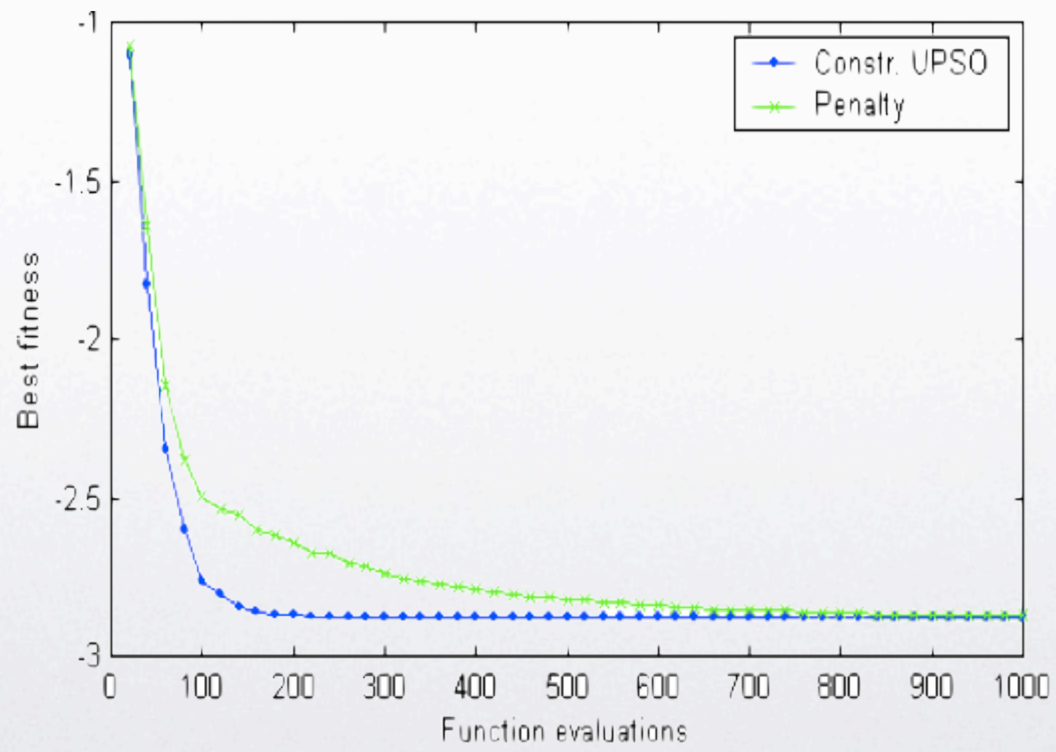
- 150 random generated problems
- $n \in [2,6]$, $N \in [3,5]$, $A_{ij} \in [-5,5]$
- every problem sampled at 0.1 resolution + local optimizer: number of local minima, one "global" minimum
- 1 to 20 local minima (single minimum in 20% of problems)
- global minimum at space boundary - 50% of problems

Numerical experiments

- For every problem 50 executions of:
  - proposed constrained UPSO (C-UPSO)
  - UPSO constrained by penalty function
- Search successful if best fitness close to global minimum

Results

- success rate: 5% higher for C-UPSO
- convergence: C-UPSO up to 50% faster
- similar influence of number of minima on both algorithms



Results

- global minimum on the boundary:
  - higher success rate of C-UPSO
  - 60% less iterations of C-UPSO

Conclusions

- proposed two modifications of PSO:
  - initialization constrained to N-1 dimensional simplex
  - ability to slide along boundary
- increase in success rate and convergence
- applicable to other eigenvalue optimization problems